$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ $E_b = \sigma T^4$

ATKINSON SCIENCE **THEORY GUIDE**

$e^{i\pi} = -1$ $\frac{u}{u_\tau} = \frac{1}{\kappa}\ln\frac{yu_\tau}{\nu} + C$

Equations of Fluid Flow

Continuity Equation in Cartesian and Cylindrical Coordinates

Keith Atkinson

10 October 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

Contents

1 Cartesian coordinates 5
2 Cylindrical coordinates 7
2.1 Method 1 7
2.2 Method 2 11
3 Integral form of the continuity equation 13
4 References 15

Figures

Figure 1 Infinitesimal control volume for Cartesian coordinates 5
Figure 2 Infinitesimal control volume for cylindrical coordinates 7

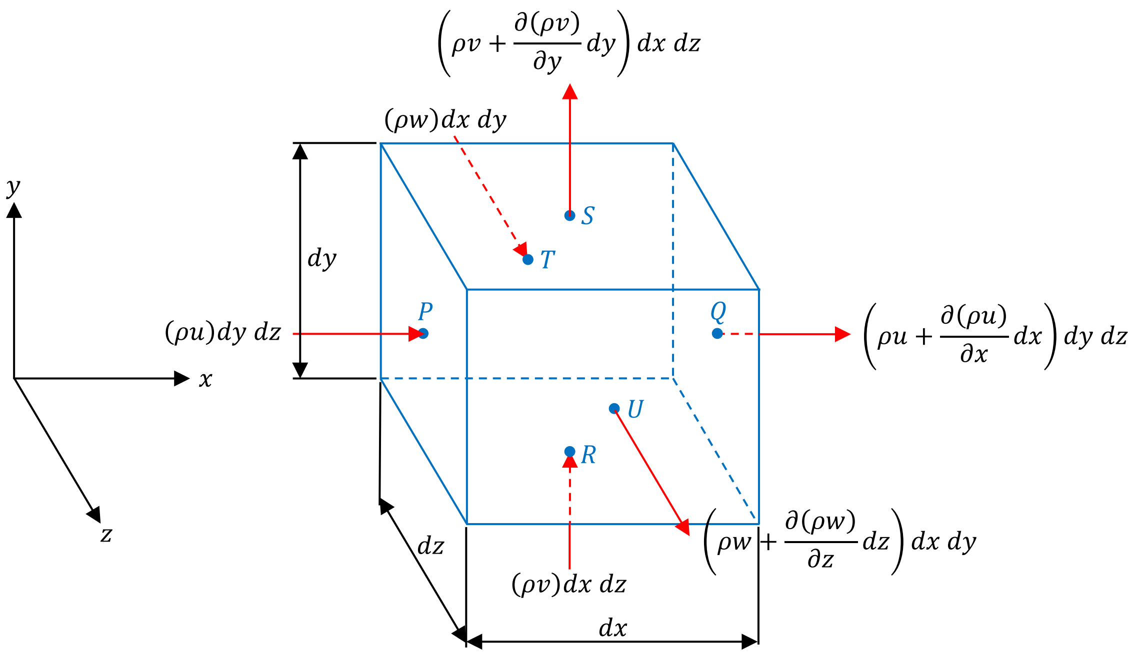
1 Cartesian coordinates

In this theory guide we shall derive the equation of conservation of mass of a moving fluid (the continuity equation) in terms of the Cartesian coordinate system and the cylindrical coordinate system. In each case the derivation is based on the concept of an infinitesimal control volume. For the case of Cartesian coordinates, the continuity equation can be derived by considering the flow of a fluid into and out of the infinitesimal rectangular control volume (CV) shown in Figure 1. The edges of the CV are parallel to the coordinate axes x, y and z, and the lengths of the edges dx, dy and dz are small enough for us to be able to neglect quantities of order dx^2, dy^2 or dz^2.

When applied to the control volume, the conservation principle for any conserved quantity Q can be written

Rate of increase of Q in the CV	=	Rate at which Q enters through the surfaces of the CV	−	Rate at which Q leaves through the surfaces of the CV	+	Sum of sources and sinks of Q in the CV		(1.1)

Figure 1 Infinitesimal control volume for Cartesian coordinates



For the continuity equation the conserved quantity Q is mass [kg], which is equal to density × volume. There are no sources or sinks of mass, so the last term in (1.1) is zero.

The mass of fluid in the CV is $\rho\ dx\ dy\ dz$ [kg] and its rate of increase with time is

$$\frac{\partial \rho}{\partial t} dx\ dy\ dz \quad [\mathrm{kg\ s^{-1}}] \quad (1.2)$$

The rate of flow of mass through the face normal to the x direction with centre P is equal to the product of the density and the velocity normal to the face (ρu) and the area of the face $dy\ dz$, i.e.

$$(\rho u) dy\ dz \quad [\mathrm{kg\ s^{-1}}]$$

The corresponding rate of flow of mass out of the parallel face with centre Q is

$$\left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dy\ dz \quad [\mathrm{kg\ s^{-1}}]$$

so the net rate of flow out of the CV through the faces with centres P and Q is

$$\left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dy\ dz - (\rho u)\ dy\ dz$$

$$= \frac{\partial(\rho u)}{\partial x} dx\ dy\ dz \quad [\mathrm{kg\ s^{-1}}] \quad (1.3)$$

Similarly, the net rate of flow out of the CV through the faces normal to the y axis with centres R and S is

$$\frac{\partial(\rho v)}{\partial y} dx\ dy\ dz \quad [\mathrm{kg\ s^{-1}}] \quad (1.4)$$

and the net rate of flow out of the CV through the faces normal to the z axis with centres T and U is

$$\frac{\partial(\rho w)}{\partial z} dx\ dy\ dz \quad [\mathrm{kg\ s^{-1}}] \quad (1.5)$$

By inserting expressions (1.2) to (1.5) into Eqn. (1.1) we obtain

$$\frac{\partial \rho}{\partial t} dx\ dy\ dz = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right) dx\ dy\ dz \quad [\mathrm{kg\ s^{-1}}]$$

After dividing by the volume of the CV, $dx\ dy\ dz$, and moving terms over to the left-hand side of the equation we obtain the continuity equation for three-dimensional unsteady compressible flow in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad [\mathrm{kg\ m^{-3}\ s^{-1}}] \quad (1.6)$$

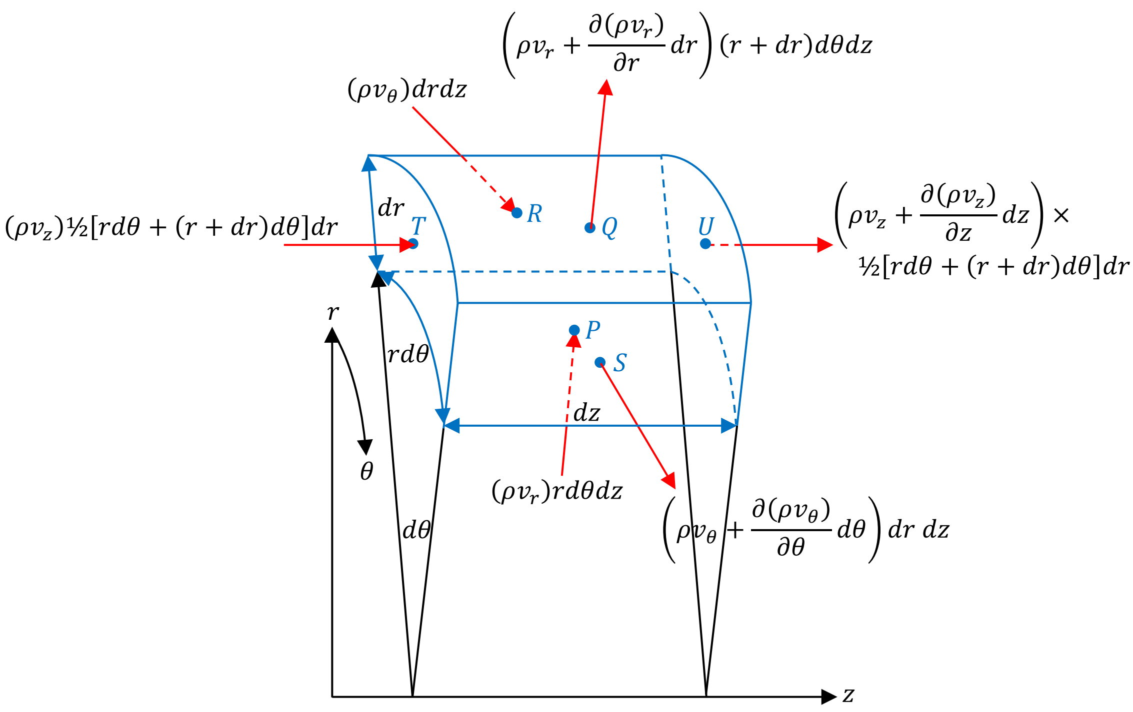
2 Cylindrical coordinates

2.1 Method 1

We can derive the continuity equation in cylindrical coordinates based on the concept of an infinitesimal control volume, just as we did with the continuity equation in Cartesian coordinates. This time we consider the properties of the flow into and out of the infinitesimal annular control volume (CV) shown in Figure 2. The lengths of the sides dr, $rd\theta$ and dz are small enough for us to be able to neglect quantities of order dr^2, $r^2d\theta^2$ or dz^2. Recalling (1.1), the conservation principle for any conserved quantity Q can be written

Rate of increase of Q in CV	=	Rate at which Q enters through the surfaces of the CV	−	Rate at which Q leaves through the surfaces of the CV	+	Sum of sources and sinks of Q within CV

Figure 2 Infinitesimal control volume for cylindrical coordinates



For the continuity equation the conserved quantity Q is mass [kg], which is equal to density × volume. There are no sources or sinks of mass, so the last term in (1.1) is zero.

The volume of the control volume is $dr\ rd\theta\ dz$ [m³], so the mass of fluid in the CV is $\rho\ dr\ rd\theta\ dz$ [kg] and its rate of increase with time is

$$\frac{\partial \rho}{\partial t} dr\ rd\theta\ dz \quad [\mathrm{kg\,s^{-1}}] \quad (2.2)$$

The rate of flow of mass through the face normal to the r direction with centre P is equal to the product of the density and the velocity normal to the face (ρv_r) and the area of the face $rd\theta\ dz$, i.e.

$$(\rho v_r)\ rd\theta\ dz \quad [\mathrm{kg\,s^{-1}}]$$

The corresponding rate of flow of mass out of the parallel face with centre Q is

$$\left(\rho v_r + \frac{\partial(\rho v_r)}{\partial r} dr\right)(r + dr)d\theta\ dz \quad [\mathrm{kg\,s^{-1}}]$$

so the net rate of flow out of the CV through the faces with centres P and Q is

$$\left(\rho v_r + \frac{\partial(\rho v_r)}{\partial r} dr\right)(r + dr)d\theta\ dz - (\rho v_r)\ rd\theta\ dz$$

$$= \left(\rho v_r + \frac{\partial(\rho v_r)}{\partial r} dr\right) rd\theta\ dz + \left(\rho v_r + \frac{\partial(\rho v_r)}{\partial r} dr\right) dr\ d\theta\ dz - (\rho v_r)\ rd\theta\ dz$$

$$= \frac{\partial(\rho v_r)}{\partial r} dr\ rd\theta\ dz + \ \rho v_r\ dr\ d\theta\ dz + \frac{\partial(\rho v_r)}{\partial r} dr^2\ d\theta\ dz \quad [\mathrm{kg\,s^{-1}}]$$

We can neglect the term in dr^2, so the net rate of flow out of the CV through the faces with centres P and Q is

$$\frac{\partial(\rho v_r)}{\partial r} dr\ rd\theta\ dz + \rho v_r\ dr\ d\theta\ dz \quad [\mathrm{kg\,s^{-1}}] \quad (2.3)$$

The rate of flow of mass through the face normal to the θ direction with centre R is equal to the product of the density and the velocity normal to the face (ρv_θ) and the area of the face $dr\ dz$, i.e.

$$(\rho v_\theta)\ dr\ dz \qquad [\mathrm{kg\ s^{-1}}]$$

The corresponding rate of flow of mass out of the face with centre S is

$$\left(\rho v_\theta + \frac{\partial(\rho v_\theta)}{\partial \theta} d\theta\right) dr\ dz \qquad [\mathrm{kg\ s^{-1}}]$$

so the net rate of flow out of the CV through the faces with centres R and S is

$$\frac{\partial(\rho v_\theta)}{\partial \theta} dr\ d\theta\ dz \qquad [\mathrm{kg\ s^{-1}}] \qquad (2.4)$$

The rate of flow of mass through the face normal to the z direction with centre T is equal to the product of the density and the velocity normal to the face (ρv_z) and the area of the face. The area of the face is

$$dr\ (r + ½dr)d\theta$$

so the rate of flow of mass through the face is

$$(\rho v_z)\ dr\ (r + ½dr)d\theta \qquad [\mathrm{kg\ s^{-1}}]$$

The corresponding rate of flow of mass out of the face with centre U is

$$\left(\rho v_z + \frac{\partial(\rho v_z)}{\partial z} dz\right) dr\ (r + ½dr)d\theta \qquad [\mathrm{kg\ s^{-1}}]$$

so the net rate of flow out of the CV through the faces with centres T and U is

$$\frac{\partial(\rho v_z)}{\partial z} dr\ (r + ½dr)d\theta\ dz\ = \frac{\partial(\rho v_z)}{\partial z} dr\ rd\theta\ dz + \frac{\partial(\rho v_z)}{\partial z} ½dr^2\ d\theta\ dz \qquad [\mathrm{kg\ s^{-1}}]$$

We can neglect the term in dr^2, so the net rate of flow out of the CV is

$$\frac{\partial(\rho v_z)}{\partial z} dr\ rd\theta\ dz \qquad [\mathrm{kg\ s^{-1}}] \qquad (2.5)$$

By inserting expressions (2.2) to (2.5) into (1.1) we obtain

$$\frac{\partial \rho}{\partial t} dr\, rd\theta\, dz = -\left(\frac{\partial(\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right) dr\, rd\theta\, dz \qquad [\mathrm{kg\ s^{-1}}]$$

After dividing by the volume of the CV, $dr\, rd\theta\, dz$ [m^3], and moving terms over to the left-hand side of the equation we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad [\mathrm{kg\ m^{-3}\ s^{-1}}]$$

We can combine the second and third terms into one, so the continuity equation for three-dimensional unsteady compressible flow in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad [\mathrm{kg\ m^{-3}\ s^{-1}}] \qquad (2.6)$$

2.2 Method 2

We can derive the continuity equation in cylindrical coordinates from the continuity equation in Cartesian coordinates by introducing the vector differential operator del, written ∇, and the divergence of a vector $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$. The vector differential operator is defined by

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

and the divergence of a vector $\mathbf{V}$ is defined by

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Further details of the vector differential operator and the divergence can be found in textbooks on vector analysis, such as Ref. [1].

We can write Eq. (1.6) as follows

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

$$= \frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial x}\mathbf{j} + \frac{\partial}{\partial x}\mathbf{k}\right)\cdot(\rho u\mathbf{i} + \rho v\mathbf{j} + \rho w\mathbf{k})$$

$$= \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0 \quad [\text{kg m}^{-3}\ \text{s}^{-1}] \quad (2.7)$$

We can express the divergence in orthogonal curvilinear coordinates as follows

$$\nabla\cdot(\rho\mathbf{V}) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(\rho A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(\rho A_3 h_1 h_2)\right] \quad (2.8)$$

(see Chapter 7 of Ref. [1]) where $(u_1,\ u_2,\ u_3)$ is an orthogonal curvilinear coordinate system, $V_1 = V_1(u_1, u_2, u_3)$, $V_2 = V_2(u_1, u_2, u_3)$, $V_3 = V_3(u_1, u_2, u_3)$, and h_1, h_2, h_3 are *scale factors*.

We can show that the cylindrical coordinate system is orthogonal (see Chapter 7, Problem 3 of Ref. [1]). For a cylindrical coordinate system, we have

$$u_1 = r,\ u_2 = \theta,\ u_3 = z$$

$$V_1 = v_r,\ V_2 = v_\theta,\ V_3 = v_z$$

$$h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_z = 1$$

(see Chapter 7, Problem 7 of Ref. [1]). Substituting these expressions into (2.8) gives

$$\nabla\cdot(\rho\mathbf{V}) = \frac{1}{r}\left[\frac{\partial(r\rho v_r)}{\partial r} + \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(r\rho v_z)}{\partial z} +\right]$$

$$= \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}$$

On substituting this expression for $\nabla \cdot (\rho \mathbf{V})$ into Eq. (2.7), we obtain the continuity equation for three-dimensional unsteady compressible flow in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \qquad [\mathrm{kg\ m^{-3}\ s^{-1}}]$$

3 Integral form of the continuity equation

The divergence theorem of Gauss states that if V is the volume bounded by a closed surface S and $\mathbf{A}(x, y, z)$ is a differentiable vector field, then

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{A}\, dV = \int_S \mathbf{A}\,.\,\mathbf{n}\, dS = \int_S \mathbf{A}\cdot d\mathbf{S} \qquad (3.1)$$

where $\mathbf{n}$ is the positive (outward drawn) normal to S. Further details of the divergence theorem and related integral theorems can be found in Ref. [1].

In Section 2 we showed how the continuity equation can be written in terms of the vector differential operator $\boldsymbol{\nabla}$ (equation (2.7)):

$$= \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{V}) = 0 \qquad [\mathrm{kg\ m^{-3}\ s^{-1}}]$$

Integrating (2.7) over a volume V inside a closed surface S gives

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\, dV = 0$$

or

$$\frac{\partial}{\partial t}\int_V \rho\, dV + \int_V \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\, dV = 0 \qquad (3.2)$$

By applying the divergence theorem to (3.2) we obtain

$$\frac{\partial}{\partial t}\int_V \rho\, dV + \int_S (\rho\mathbf{v}).\mathbf{n}\, dS = 0 \qquad (3.3)$$

We now have the equation of continuity in terms of the vector differential operator $\boldsymbol{\nabla}$ (2.7) and as an integral equation (3.3).

4 References

1. M. R. Spiegel, *Vector Analysis and an Introduction to Tensor Analysis*, Schaum's Outline Series, McGraw-Hill, 1959.